

做试卷要点

空间应力状态 / 平面应力状态

广义胡克定律

$$\epsilon_1 = \frac{1}{E} [\sigma_1 - \mu(\sigma_2 + \sigma_3)]$$

$$\epsilon_2 = \frac{1}{E} [\sigma_2 - \mu(\sigma_1 + \sigma_3)]$$

$$\epsilon_3 = \frac{1}{E} [\sigma_3 - \mu(\sigma_1 + \sigma_2)]$$

强度理论: $\sigma_1 > \sigma_2 > \sigma_3$ 主应力

$$\sigma_{r1} = \sigma_{\max}$$

$$\sigma_{r2} = \sigma_1 - \mu(\sigma_2 + \sigma_3)$$

$$\sigma_{r3} = \sigma_1 - \sigma_3$$

$$\sigma_{r4} = \sqrt{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2]}$$

交变应力

→ 截面方向 $M_{\max}$

扭矩 弯矩, 正应力, 切应力

↓ 沿杆方向 $M_{\max}$ 由 $T_{\max}$

期末大题:

一: 应力状态分析

二: 组合变形

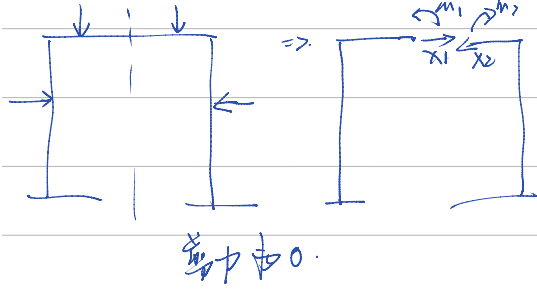
三: 压杆稳定

四: 能量法估超静定

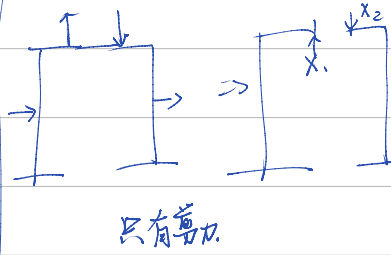
$$\begin{aligned}\sigma_{r4} &= \sqrt{\frac{1}{2}[(\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_2)^2]} \\ &= \sqrt{\sigma^2 + 3\tau^2} \\ &= \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2}\end{aligned}$$

对称荷载与反对称

$F_{S=0}$



$M=0 \quad F_N=0$



求解图形对称结构:

公式整理:

基本变形:

$$\left\{ \begin{array}{l} \text{拉压: } \sigma = \frac{F_N}{A}, \quad \Delta L = \frac{F_N L}{EA}, \quad \epsilon = \frac{\sigma}{E} = \frac{F_N}{EA} \\ \text{扭转: } \tau = \frac{T \rho}{I_P}, \quad \varphi = \frac{T L}{G I_P}, \quad \gamma = \frac{\tau}{G} \text{ (切应变)} \\ \text{弯曲: } \sigma = \frac{M y}{I_z} \text{ (弯曲正应力)} \\ \tau = \frac{F_s S_z^*}{I_z b} \text{ (弯曲切应力)} \Rightarrow \tau_{\max, \text{矩}} = \frac{3}{2} \bar{\tau} = \frac{3}{2} \frac{F_s}{A} \\ \tau_{\max, \text{圆}} = \frac{4}{3} \bar{\tau} \end{array} \right.$$

惯性矩



$$I_z = \frac{1}{12} b h^3$$


$$I_z = \frac{1}{64} \pi d^4, \quad I_P = \frac{1}{32} \pi d^4$$

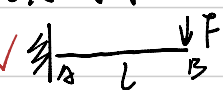
② $I_z = \frac{1}{64} \pi (D^4 - d^4)$
 $= \frac{1}{64} \pi b^4 (1 - \alpha^4)$

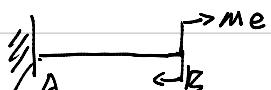
弯曲变形:

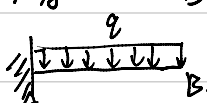
欧拉经典梁理论 $EI w'''' = -M(x), \quad \theta = w'$
 $EI w'''' = f(x), \quad EI w''' = q(x)$

挠度: 简单几个, 可用能量法做

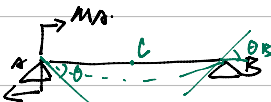
① 悬臂梁:

✓  $w_B = \frac{1}{3} \frac{F l^3}{EI}, \quad \theta_B = \frac{1}{2} \frac{F l^2}{EI}$

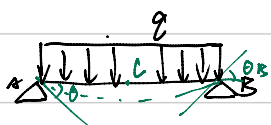
 $w_B = \frac{1}{2} \frac{M l^2}{EI}, \quad \theta_B = \frac{M l}{EI}$

 $w_B = \frac{1}{8} \frac{q l^4}{EI}, \quad \theta_B = \frac{1}{6} \frac{q l^3}{EI}$

② 简支梁:

 $\theta_A = \frac{M l}{3EI}, \quad \theta_B = -\frac{M l}{6EI}, \quad w_C = \frac{1}{16} \frac{M l^2}{EI}$

✗  $\theta_A = \frac{1}{16} \frac{F l^2}{EI}, \quad \theta_B = -\frac{1}{16} \frac{F l^2}{EI}, \quad w_C = \frac{1}{48} \frac{F l^3}{EI}$

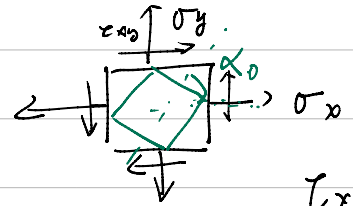
 $\theta_A = \frac{1}{24} \frac{q l^3}{EI}, \quad \theta_B = -\frac{1}{24} \frac{q l^3}{EI}, \quad w_C = \frac{5}{384} \frac{q l^4}{EI}$

③ 外伸梁 别说了.



应力状态分析:

$$\sigma_1 > \sigma_2 > \sigma_3$$



$$\sigma_\alpha = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\alpha - \tau_{xy} \sin 2\alpha$$

$$\tau_\alpha = \frac{\sigma_x - \sigma_y}{2} \sin 2\alpha + \tau_{xy} \cos 2\alpha \quad \text{主应力方向角 } \tan 2\alpha_0 = -\frac{\tau_{xy}}{\sigma_x - \sigma_y}$$

主应力:

$$\sigma_{\max, \min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

强度理论:

$$\text{广义胡克定律: } \epsilon_i = \frac{1}{E} [\sigma_i - \mu(\sigma_2 + \sigma_3)]$$

相当应力

$$\sigma_{r1} = \sigma_1$$

$$\sigma_{r2} = \sigma_1 - \mu(\sigma_2 + \sigma_3)$$

$$\sigma_{r3} = \sigma_1 - \sigma_3 \xrightarrow{\text{平面}} \sqrt{\sigma^2 + 4\tau^2}$$

$$\sigma_{r4} = \sqrt{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2]} \Rightarrow \sqrt{\sigma^2 + 3\tau^2}$$

组合变形

开槽柱

偏心压缩 截面核心

中性轴方程

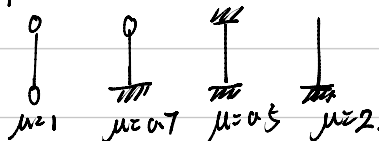
压杆稳定:

$$\text{柔度: } \lambda = \frac{\mu l}{i} \quad i = \sqrt{\frac{I}{A}}$$

$$\lambda_p = \pi \sqrt{\frac{E}{\sigma_p}} \rightarrow \text{比例极限}$$

$$\lambda_s = \pi \sqrt{\frac{E}{\sigma_s}} \rightarrow \text{屈服极限}$$

$$\text{大柔度 } \lambda > \lambda_p \Rightarrow \text{For} = \frac{\pi^2 EI}{(\mu l)^2}$$



$$\text{中: } \lambda_0 < \lambda_p \Rightarrow \sigma_{or} = a - b\lambda$$

$$\text{小: } \lambda < \lambda_s \Rightarrow \sigma < \sigma_s$$

动载荷:

动荷系数: $k_d = 1 + \sqrt{1 + \frac{2h}{\sigma_{st}}}$

$$\Delta d = k_d \cdot \sigma_{st}$$

$$\sigma_d = k_d \cdot \sigma_{st}$$

交变应力:

循环特征: $r = \frac{\sigma_{min}}{\sigma_{max}}$

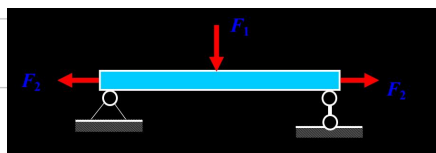
平均应力: $\sigma_m = \frac{1}{2} (\sigma_{max} + \sigma_{min})$

应力幅: $\sigma_a = \frac{1}{2} \sigma_{max} - \sigma_{min}$

组合变形:

① 双弯组合变形.

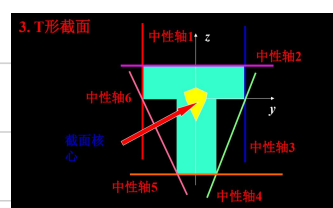
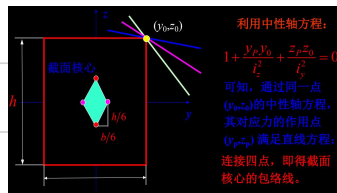
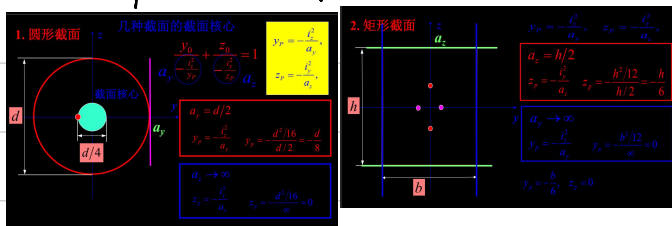
② 拉压弯组合.



$$\sigma_1 = \frac{F_N}{A} \quad \sigma_2 = \frac{M_{max}}{W}$$

$$\sigma = \frac{F_N}{A} + \frac{M_{max}}{W}$$

③ 偏心压缩. (截面核心)

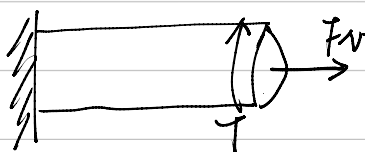


中性轴方程: $\sigma_x = \frac{F}{A} + \frac{F y_0 y}{I_z} + \frac{F z_0 z}{I_y} = 1 + \frac{y_0 y}{i_z^2} + \frac{z_0 z}{i_y^2} = 0$

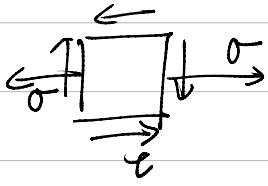
$$\Leftrightarrow \frac{y_0}{-\frac{i_z^2}{y_0}} + \frac{z_0}{-\frac{i_y^2}{z_0}} = 1$$

$\xrightarrow{y_0} a_y \quad \xrightarrow{z_0} a_z$

④ 拉扭组合.



⑤ 弯扭组合:



$$\sigma_{r3} = \sqrt{\sigma^2 + 4\tau^2}$$

$$\begin{aligned}\sigma_{r4} &= \sqrt{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2]} \\ &= \sqrt{\sigma^2 + 3\tau^2}\end{aligned}$$

压杆稳定:

① 柔度: $\lambda = \frac{\mu l}{i}$ $i = \sqrt{\frac{I}{A}}$ 回转半径.

$\lambda = \pi \sqrt{\frac{E}{\sigma_p}}$

直线公式: $\sigma_{cr} = a - b\lambda$.

② 欧拉临界力公式:

$F_{cr} = \frac{\pi^2 EI}{(\mu l)^2}$

1. 两端铰支, $\mu=1$
2. 一端铰支, 一端固定, $\mu=0.7$
3. 两端固定, $\mu=0.5$
4. 一端固定, 一端自由, $\mu=2$

各种支承约束条件下等截面细长压杆临界力的欧拉公式 P309页 表9-1

支承情况	两端铰支	一端固定 另一端铰支	两端固定	一端固定 另一端自由	两端固定但可沿 横向相对移动
失稳时挠曲线形状					
临界力 F_{cr} 欧拉公式	$F_{cr} = \frac{\pi^2 EI}{l^2}$	$F_{cr} = \frac{\pi^2 EI}{(0.7l)^2}$	$F_{cr} = \frac{\pi^2 EI}{(0.5l)^2}$	$F_{cr} = \frac{\pi^2 EI}{(2l)^2}$	$F_{cr} = \frac{\pi^2 EI}{l^2}$
长度系数 μ	$\mu=1$	$\mu=0.7$	$\mu=0.5$	$\mu=2$	$\mu=1$

③ 直线经验公式

$\sigma_{cr} = a - b\lambda$



解题步骤: ① 根据 $\lambda = \pi \sqrt{\frac{E}{\sigma}}$ σ_s, σ_p
求 λ_s, λ_p .

② $\lambda = \frac{\mu l}{i}$ $i = \sqrt{\frac{I}{A}}$ 求 λ

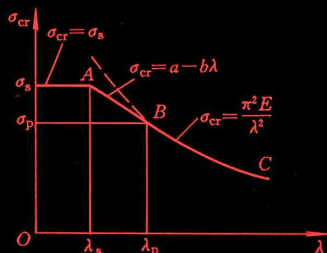
判断: $\lambda > \lambda_p$: 大柔度 $\rightarrow F_{cr} = \frac{\pi^2 EI}{(\mu l)^2}$

$\lambda_s < \lambda < \lambda_p$: 中柔度 $\rightarrow \sigma_{cr} = a - b\lambda$

$\lambda < \lambda_s$: 小柔度杆 $\rightarrow$ 校核强度 σ_s .

2. 经验公式—直线公式

$\sigma_{cr} = a - b\lambda$ (a, b 是与材料性质有关的常数)



压杆的临界应力总图

$\lambda \geq \lambda_p$ ($\sigma \leq \sigma_p$) (大柔度杆)

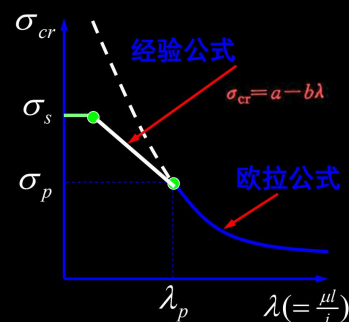
欧拉公式

$\sigma_p \leq \sigma \leq \sigma_s$ (中柔度杆)

经验公式

$\sigma = \sigma_s$ (小柔度杆)

强度问题



压杆的临界应力总图

能量方法

功能互等: $V_e = W$

线弹性: 应变能

1. 拉伸:
$$\begin{cases} W = \frac{1}{2} F \Delta l \\ \Delta l = \frac{FL}{EA} \end{cases} \Rightarrow V_e = \frac{FL^2}{2EA}$$

2. 扭转:
$$\begin{cases} W = \frac{1}{2} T \varphi \\ \varphi = \frac{TL}{GI_p} \end{cases} \Rightarrow V_e = \frac{TL^2}{2GI_p}$$

3. 弯曲:
$$\begin{cases} W = \frac{1}{2} M \theta \\ \theta = \frac{\Delta}{\rho} = \frac{M \Delta}{EI} \end{cases} \Rightarrow V_e = \frac{M \Delta^2}{2EI}$$

应变能 $V_e = \int_0^{\epsilon} V_e \cdot d\epsilon \rightarrow \begin{cases} V_e = \frac{1}{2} \sigma \epsilon \\ V_e = \frac{1}{2} E \epsilon^2 \end{cases}$

注意: 应变能一般不可叠加。
分析时不要把中算为杆。

$$V_e = \int_L \frac{F_N^2 \Delta x}{2EA} \cdot dx + \int_L \frac{T^2 \Delta x}{2GI_p} \cdot dx + \int_L \frac{M^2 \Delta x}{2EI} \cdot dx$$

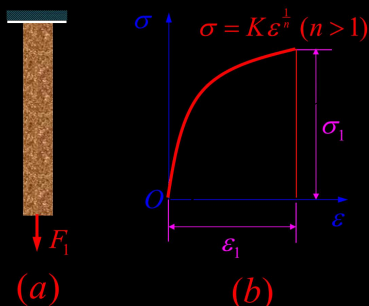
功的互等定理

$$P_1 \Delta_{12} = P_2 \Delta_{21}$$

用角不多。

非线性弹性:

例4: 计算图 (a) 杆在 F_1 轴力作用下的应变能。杆的长度为 l , 截面积为 A , 应力-应变曲线如图 (b)。



$$\begin{aligned} V_e &= \int_0^{\epsilon_1} V_e \cdot d\epsilon = \int_0^{\epsilon_1} K \cdot \epsilon^{\frac{1}{n}} \cdot d\epsilon \\ &= \frac{n}{n+1} \cdot K \cdot \epsilon^{\frac{n+1}{n}} \Big|_0^{\epsilon_1} \\ &= \frac{n}{n+1} K \cdot \epsilon_1^{\frac{n+1}{n}} \end{aligned}$$

计算总能 $\Rightarrow V_e = \int_0^{\epsilon_1} \epsilon \cdot d\sigma$

卡氏定理:



概况:

广义力:

广义位移.

卡一: $F_i = \frac{\partial V_e(\Delta_1, \Delta_2, \dots, \Delta_n)}{\partial \Delta_i}$

卡二: $\Delta_i = \frac{\partial V_e(F_1, F_2, \dots, F_n)}{\partial F_i}$

(求能量法解线弹性系统变式)

$$V_e = \int_L \frac{F_N^2(x)}{2EA} \cdot dx + \int_L \frac{M^2(x)}{2EI} \cdot dx + \int_L \frac{T^2(x)}{2GI_p} \cdot dx.$$

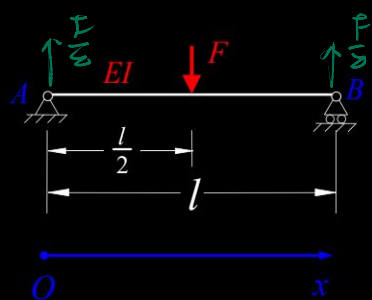
解题步骤:

- ① 看约束, 解中自条件
- ② 施加广义力/力偶
- ③ 写出各段弯矩
- ④ 写应变能 $\rightarrow$ 卡二 $\rightarrow$ 先偏导后积分.

挠度可用能量法推

例2: 求简支梁中点作用集中力 F 作用时中点处的挠度。

(梁的弯曲刚度为 EI , 长为 l)。



解: 先求应变能

$$V_e = \int_0^l \frac{M^2(x)}{2EI} dx = 2 \times \int_0^{l/2} \frac{(\frac{F}{2}x)^2}{2EI} dx$$

$$V_e = \frac{F^2 l^3}{96EI}$$

$$w_c = \frac{\partial V_e}{\partial F} = \frac{Fl^3}{48EI}$$

$$M(x) = \frac{F}{2} \cdot x,$$

$$V_e = 2 \int_0^{l/2} \frac{(\frac{F}{2} \cdot x)^2}{2EI} \cdot dx. \quad w = \frac{\partial V_e}{\partial F} = \frac{1}{EI} \int_0^{l/2} \frac{F}{2} x^2 \cdot dx = \frac{Fl^3}{48EI}$$

求得自正负代表
与广义力偶方向正负。

解(2): 求转角 $\theta = \frac{\partial V_\varepsilon}{\partial M}$
假想在自由端作用一力矩 M

$\theta = \frac{\partial V_\varepsilon}{\partial M} \Big|_{M=0}$

$V_\varepsilon = \int_0^l \frac{M^2(x)}{2EI} dx = \int_0^l \frac{[-(Fx + M)]^2}{2EI} dx$

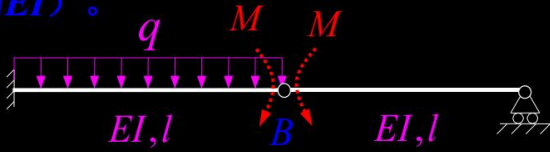
接下来先完成求导工作，
不要去做积分!!!

$\theta = \frac{\partial V_\varepsilon}{\partial M} \Big|_{M=0} = \int_0^l \frac{-(Fx + M)}{EI} \cdot (-1) dx \Big|_{M=0}$

下面先完成 $M=0$ 的代入，进而完成积分!

$= \int_0^l \frac{Fx}{EI} dx = \frac{Fl^2}{2EI}$

例6: 求铰B两侧截面的相对转角 (各杆的弯曲刚度为 EI)。

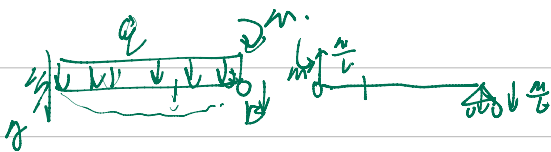
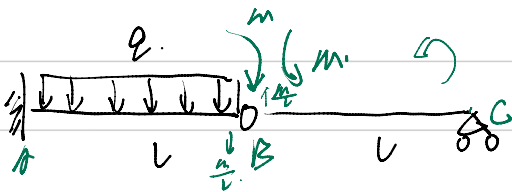


解: 在铰B处施加一对力矩 M , 则

$$\theta_B = \frac{\partial V_\varepsilon}{\partial M} \Big|_{M=0}$$

施加一对力偶

$\frac{\partial V_\varepsilon}{\partial m}$ 即两转角之和。



$$M_{AB}(x) = -m - \frac{m}{l} \cdot x - \frac{1}{2} q x^2$$

$$M_{BC}(x) = -\frac{m}{l} \cdot x$$

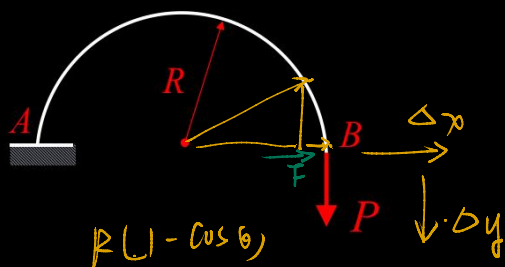
$$\frac{\partial M_{AB}}{\partial m} = -1 - \frac{x}{l} \quad \frac{\partial M_{BC}}{\partial m} = -\frac{x}{l}$$

$$\begin{aligned} \therefore \theta &= \frac{\partial V_\varepsilon}{\partial m} \Big|_{m=0} = \frac{1}{EI} \int_0^l M_{AB} \frac{\partial M_{AB}}{\partial m} dx + \int_0^l M_{BC} \frac{\partial M_{BC}}{\partial m} dx \\ &= \frac{1}{EI} \cdot \int_0^l \left(m + \frac{m}{l} x + \frac{1}{2} q x^2 \right) \left(1 + \frac{x}{l} \right) dx + \int_0^l m \cdot \left(\frac{x}{l} \right)^2 dx \\ &= \frac{1}{EI} \int_0^l \left(\frac{1}{2} q x^2 + \frac{1}{2} q x^3 / l \right) dx = \frac{7 q l^3}{24 EI} \end{aligned}$$

曲杆问题:

不太会

例7: 求图示半圆形曲杆B点的水平位移、铅直位移 (不计轴力及剪力对位移的影响)。



$$V_E = \int \frac{M^2}{2EI} \cdot dx$$

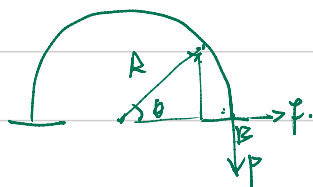
$$\Downarrow V_E = \int \frac{M^2}{2EI} \cdot ds = \int \frac{M^2}{2EI} R \cdot d\theta$$

$$M(\theta) = -PR(1 - \cos\theta)$$

$$V_E = \int_0^\pi \frac{M^2}{2EI} R d\theta = \int_0^\pi \frac{P^2 R^3 (1 - \cos\theta)^2}{2EI} \cdot d\theta$$

$$\Delta y = \frac{\partial V_E}{\partial P} = \int_0^\pi \frac{PR^3 (1 - \cos\theta)^2}{EI} \cdot d\theta = \frac{PR^3}{EI} \left(\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta \right) \Big|_0^\pi = \frac{5\pi PR^3}{2EI} \quad (\downarrow)$$

求 Δx . 向右施加单位力



$$M_1 = FR \sin\theta$$

$$M_2 = -PR(1 - \cos\theta)$$

$$\Delta M = FR \sin\theta - PR(1 - \cos\theta)$$

$$\frac{\partial M}{\partial F} = R \sin\theta$$

$$\therefore \Delta x = \frac{\partial V_E}{\partial F} \Big|_{F=0} = \frac{R}{EI} \int_0^\pi -PR(1 - \cos\theta) \cdot R \sin\theta \cdot d\theta$$

$$= \frac{R}{EI} \int_0^\pi -PR^2 \sin\theta (1 - \cos\theta) \cdot d\theta$$

$$= \frac{-PR^3}{EI} \left(-\cos\theta + \frac{1}{4}\cos 2\theta \right) \Big|_0^\pi = \frac{-2PR^3}{EI}$$

单位载荷法 / 莫尔积分

能量法掌握一种即可。

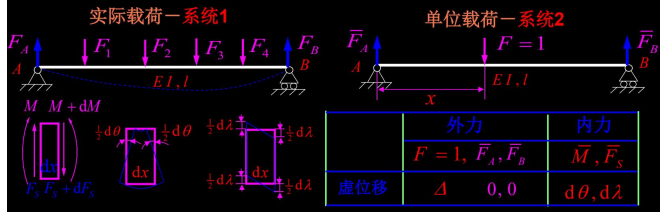
载荷作用

单位力作用

虚功原理

$$\sum_{i=1}^n F_i \bar{\Delta}_i = \int_l (M d\theta + \bar{F}_S \cdot d\lambda + \bar{F}_N d\delta + \bar{T} d\varphi)$$

单位载荷法(对系统2) $1 \cdot \Delta = \int_l (\bar{M} d\theta + \bar{F}_S \cdot d\lambda + \bar{F}_N d\delta + \bar{T} d\varphi)$



$$\Delta = \int_l \frac{M \cdot \bar{m}}{EI} \cdot dx + \int_l \frac{F_N \bar{F}_N}{EI} \cdot dx + \int_l \frac{T \cdot \bar{T}}{GI_p} \cdot dx$$

单位载荷法 $1 \cdot \Delta = \int_l (\bar{M} d\theta + \bar{F}_S \cdot d\lambda + \bar{F}_N d\delta + \bar{T} d\varphi)$

推导单位载荷法时，将虚设的单位力作为载荷，将实际载荷引起的位移当作虚位移。

对于线弹性杆件，其虚位移有：大于1的因素，与截面形状有关

$$d\delta = \frac{F_N dx}{EA}, \quad d\theta = \frac{M dx}{EI}, \quad d\lambda = \frac{\alpha_s F_s dx}{GA}, \quad d\varphi = \frac{T dx}{GI_p} \quad (1)$$

$$1 \cdot \Delta = \int_l (\bar{M} \frac{M}{EI} dx + \bar{F}_S \cdot \frac{\alpha_s F_s}{GA} dx + \bar{F}_N \frac{F_N}{EA} dx + \bar{T} \frac{T}{GI_p} dx)$$

注意：因导出上式的单位载荷法公式用了仅适用于线弹性杆的式(1)，因此上式仅适用于线弹性杆件。

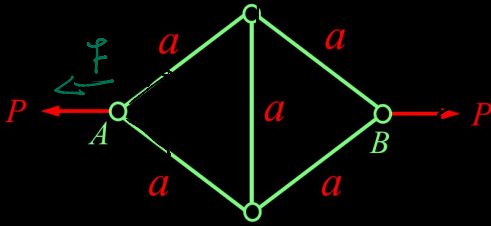
$$1 \cdot \Delta = \int_l (\bar{M} \frac{M}{EI} dx + \bar{F}_S \cdot \frac{\alpha_s F_s}{GA} dx + \bar{F}_N \frac{F_N}{EA} dx + \bar{T} \frac{T}{GI_p} dx)$$

应用上式计算位移时需注意：

1. 右端的项目根据具体情况而定，不一定每项都有
2. 单位力是广义力，视需确定的位移而定
3. 计算结果为正，说明所求位移与单位力指向一致
计算结果为负，说明所求位移与单位力指向相反
4. 公式右端各项内力的正负号规定同前
5. 积分遍布整个结构
6. 处理桁架问题时，公式右端只剩轴力这一项
7. 也统称为莫尔定理，积分称为莫尔积分

桁架问题：

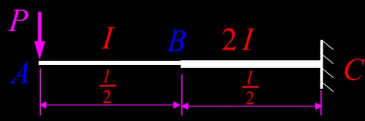
例1：用单位载荷法求AB间的相对位移（各杆的拉压刚度为EA，长度均为a）



解：A、B点施加单位力F

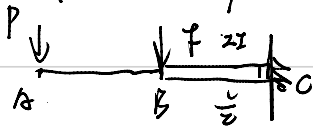
$$\Delta_{AB} = \int \frac{F \cdot \bar{F}}{EA} \cdot dx = 5 \cdot \frac{P}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \cdot \frac{a}{EA} = \frac{5Pa}{3EA}$$

例3: 用单位载荷法求图示梁B点的挠度和A点的转角(不计剪切变形的影响, 梁材料的弹性模量为E)。



解:

① 求B点挠度: 在B处施加向下单位力



纯弯。

$$\Delta = \int_L \frac{m \cdot \bar{m}}{EI} \cdot dx + \int_L \frac{m \cdot \bar{m}}{2EI} \cdot dx$$

AB段: $M(x) = P \cdot x, 0 \leq x \leq \frac{l}{2}$

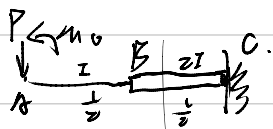
$\bar{m}(x) = 0$

BC段: $M(x) = P \cdot x, \frac{l}{2} \leq x \leq l$

$\bar{m}(x) = x - \frac{l}{2}, \frac{l}{2} \leq x \leq l$

$$\therefore \Delta = \int_{\frac{l}{2}}^l \frac{Px(x - \frac{l}{2})}{2EI} \cdot dx = \frac{P}{2EI} \left(\frac{1}{3}x^3 - \frac{1}{2}lx^2 \right) \Big|_{\frac{l}{2}}^l = \frac{5P}{96EI}$$

② 求A点转角: 在A处施加单位力偶 M_0



AB: $M(x) = Px, 0 \leq x \leq \frac{l}{2}$ BC: $M(x) = Px$

$\bar{m} = m_0 = 1$

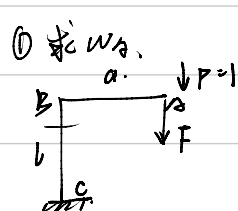
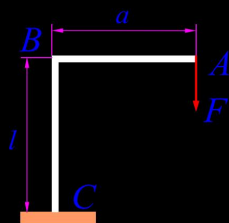
$\bar{m} = m_0 = 1$

$$\therefore \theta = \int_0^{\frac{l}{2}} \frac{(Px)}{EI} \cdot dx + \int_{\frac{l}{2}}^l \frac{(Px)}{2EI} \cdot dx$$

$$= \frac{Px^2}{2EI} \Big|_0^{\frac{l}{2}} + \frac{Px^2}{4EI} \Big|_{\frac{l}{2}}^l = \frac{5Pl^2}{16EI} \quad (\text{逆时针})$$

刚架问题.

例4: 图示刚架的自由端A作用集中载荷F。刚架各段的抗弯刚度为EI。若不计轴力和剪力对位移的影响, 试用单位载荷法计算A点的垂直位移 w_A 及截面B的转角 θ_B 。



① 求 w_A .

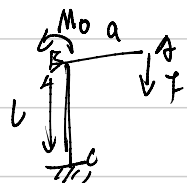
AB段: $M(x) = -Fx$ $\bar{M}(x) = -x$ $0 < x < a$

BC段: $M(x) = -Fa$ $\bar{M}(x) = -a$ $0 < x < l$

$$\therefore w_A = \int_0^a \frac{Fx^2}{EI} dx + \int_0^l \frac{Fa^2}{EI} dx$$

$$= \frac{F}{EI} \cdot \frac{1}{3} x^3 \Big|_0^a + a^2 x \Big|_0^l = \frac{Fa^2}{3EI} (a + 3l)$$

② 求 θ_B . 在B处施加 M_0



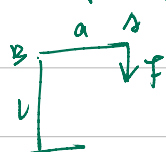
AB: $M(x) = -Fx$ $\bar{M} = 0$

BC段: $M(x) = -Fa$ $\bar{M} = M_0$

$$\therefore \theta_B = \int_0^l \frac{-Fa}{EI} \cdot dx = \frac{-Fal}{EI}$$

$$\therefore \theta_B = \frac{Fal}{EI} \text{ (顺时针)}$$

如果用卡二做:



AB: $M(x) = -Fx$ BC: $M(x) = -Fa$

$$w_A = \frac{\partial V_0}{\partial F} = \frac{1}{EI} \left(\int_0^a \frac{Fx^2}{EI} + \int_0^l \frac{Fa^2}{EI} \right) = \frac{Fa^2}{3EI} (a + 3l)$$



AB: $M(x) = -Fx$ BC: $M(x) = -Fa + M_0$

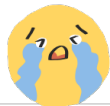
$$\theta_B = \frac{\partial V_0}{\partial M_0} \Big|_{M_0=0} = \frac{1}{EI} \int_0^l -Fa \cdot dx = \frac{-Fal}{EI}$$

单位卡二 / 卡二 做题观感差不多.

图乘法算了.



超静定问题



能量法求解步骤:

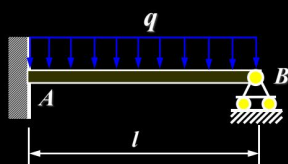
① 看超静定个数, 列平衡方程.

② 找几何条件. 解除约束, 用能量法列变形协调方程 作为补充方程 (一般 $\Delta = 0$, $\theta = 0$).

③ 求解未知力. 力偶

$$\Delta_1 + \Delta_2 = 0$$

例题1 如图所示, 梁EI为常数, 试求支座反力.

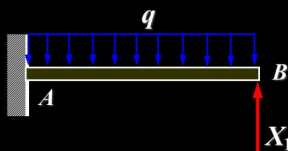


(1) 去掉多余约束代之约束反力, 得基本静定系

把 B 支座作为多余约束

AB 悬臂梁为基本静定系

X_1 为多余反力



中弦正则方程.

zwp 不讲.

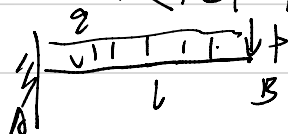
更好玩的是: 直接卡二

去除 B 约束代以反力 X_1

$$W_B + \Delta X_1 = 0$$

求 W_B :

B 处施加单位力 $P=1$



$$AB: M(x) = \frac{1}{2} q x^2 \quad \bar{m} = x$$

$$W_B = \int_0^l \frac{\frac{1}{2} q x^2 \cdot x}{EI} \cdot dx = \frac{q l^4}{8 EI} \quad (1)$$



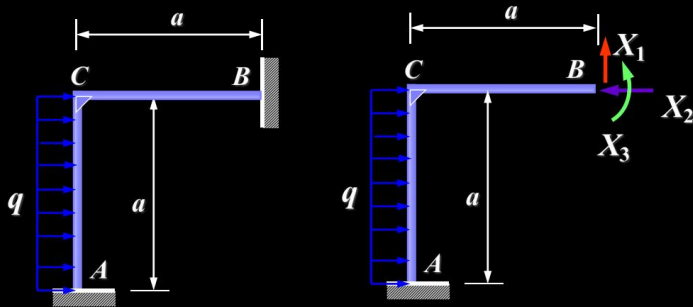
$$\Delta X_1 = \frac{\partial V_G}{\partial X_1} = \frac{\partial \int_0^l \frac{X_1^2 x^2}{2 EI} \cdot dx}{\partial X_1}$$

$$= \int_0^l \frac{X_1 x^2}{EI} \cdot dx = \frac{X_1 l^3}{3 EI}$$

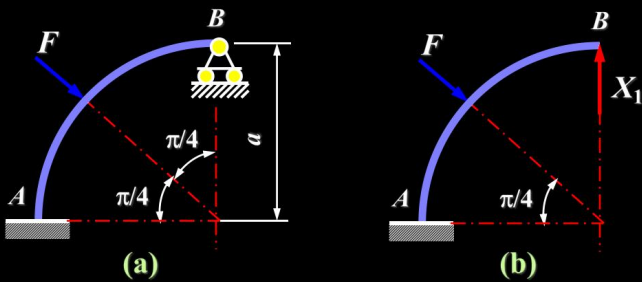
$$\Rightarrow \frac{X_1 l^3}{3 EI} + \frac{q l^4}{8 EI} = 0 \quad \therefore X_1 = -\frac{3}{8} q l \quad (\uparrow)$$

再找点题做

例题 求解静不定结构刚架,设两杆的 EI 相等.



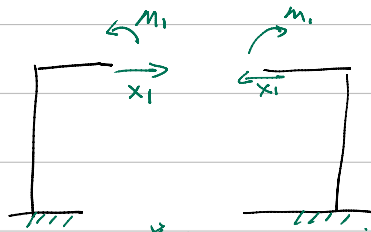
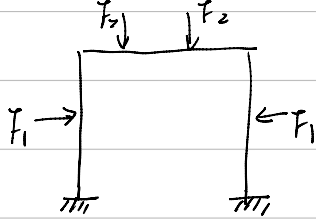
例题 轴线为四分之一圆周的曲杆 A 端固定, B 端铰支(图a).在 F 作用下,试求曲杆的弯矩图.设曲杆横截面尺寸远小于轴线半径,可以借用计算直杆变形的公式.



解:曲杆为一次超静定,解除多余与支座 B ,得到 A 端固定, B 端为自由端的基本静定系,多余约束力为 X_1 (图b).

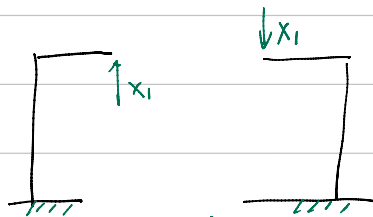
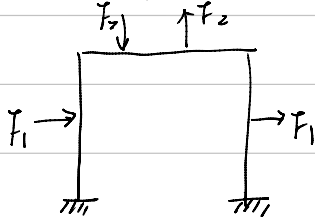
对称 / 反对称

1> 对称荷载



无剪力, 有轴力, 弯矩.

2> 反对称荷载



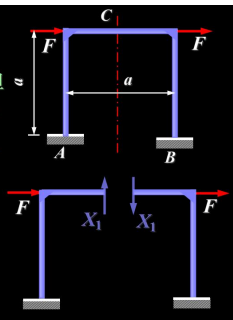
仅有剪力.

例题 试求图示刚架的全部约束反力, 并作弯矩图, 刚架EI为常数.

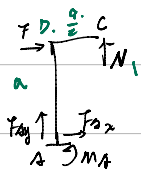
解: 图示刚架有三个多余未知力. 但由于结构是对称的, 而荷载反对称, 故对称轴横截面上轴力, 弯矩为零, 只有一个多余未知力 (剪力), 只需列出一个正则方程求解.

$$\delta_{11} X_1 + \Delta_{1F} = 0$$

用莫尔定理求 Δ_{1F} 和 δ_{11} .



解: 沿对称中心截断分析.



$$\begin{cases} \sum F_y = 0 \\ \sum F_x = 0 \\ \sum M_A = 0 \end{cases} \Rightarrow \begin{cases} F_{Ay} = -N_1 \\ F_{Ax} = -F \\ M_A = aF - \frac{a}{2} N_1 \end{cases}$$

$$\Rightarrow \text{求 } N_1 \quad \Delta C = 0 \quad \frac{\partial V_C}{\partial N_1} = 0$$

$$CD: M(x) = N_1 x, \quad 0 < x < \frac{a}{2} \Rightarrow \frac{\partial M}{\partial N_1} = x$$

$$AB: M(x) = \frac{a}{2} N_1 - Fx, \quad 0 < x < a \quad \frac{\partial M}{\partial N_1} = \frac{a}{2}$$

$$\Rightarrow \Delta x = \frac{\partial V_C}{\partial N_1} = \frac{1}{EI} \left(\int_0^{\frac{a}{2}} N_1 x^2 dx + \int_0^a \left(\frac{a}{2} N_1 - Fx \right) \frac{a}{2} dx \right)$$

$$= \frac{1}{EI} \left(\frac{1}{3} N_1 x^3 \Big|_0^{\frac{a}{2}} + \frac{a^2}{4} N_1 x - \frac{aF}{4} x^2 \Big|_0^a \right)$$

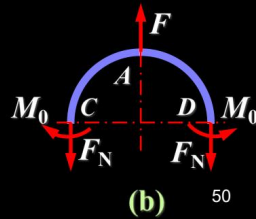
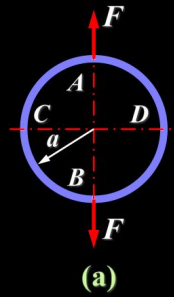
$$= \frac{1}{EI} \left(\frac{1}{24} N_1 a^3 + \frac{1}{4} N_1 a^2 - \frac{1}{4} F a^2 \right) = 0 \quad \therefore N_1 = \frac{6}{7} F$$

解完了.

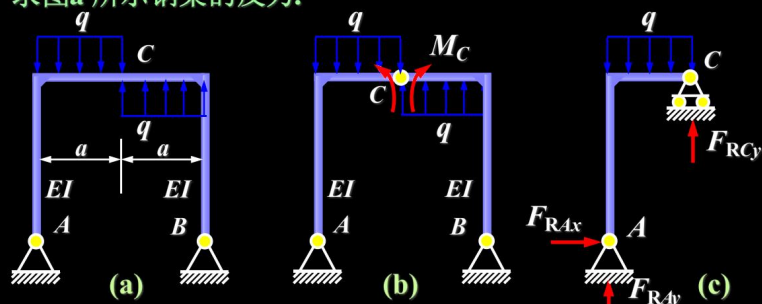
例题 在等截面圆环直径 AB 的两端,沿直径作用方向相反的一对 F 力 (图a) .试求 AB 直径的长度变化.

解:沿水平直径将圆环切开 (图b) ,由载荷的对称性,截面 C 和 D 上的剪力等于零,只有轴力 F_N 和弯矩 M_0 .

利用平衡条件求出 $F_N=F/2$,只有 M_0 为多余约束力.



例题 求图a所示钢架的反力。



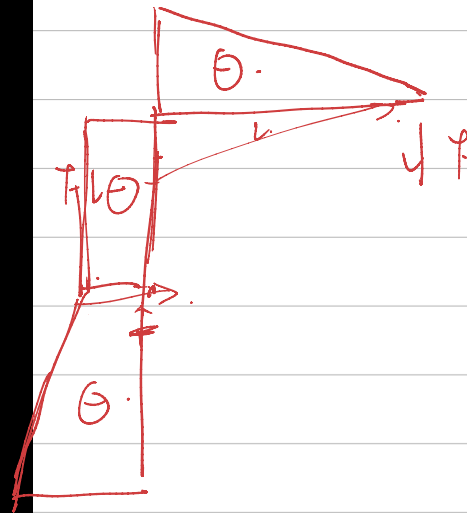
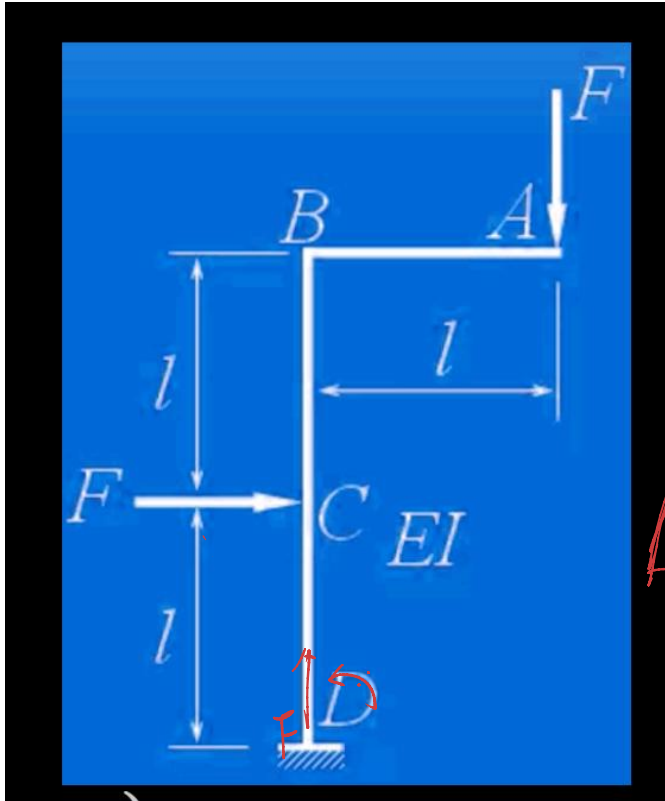
解:钢架有4个反力,是一次超静定结构。基本静定系如图b。
结构上的载荷是反对称的,C截面只有剪力不为零。

钢架的左半部分简化成图(c)所示情况,由平衡方程得到:

$$F_{R4y} = \frac{qa}{2} \quad F_{RCy} = \frac{qa}{2} \quad F_{R4x} = 0$$

卡二求各截面的铅垂位移和水平位移

钢架



曲杆

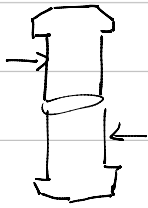
2017 期末复习课.

低碳钢拉伸应力-应变曲线

屈服极限.

弹性模量.

连接件计算



$$\tau = \frac{F_s}{A}$$

$$\sigma_{bs} = \frac{F_{bs}}{A_{bs}} \rightarrow \text{直径投影面}$$

高压圆筒薄壁.

轴向: $F = p \cdot \frac{\pi D^2}{4}$

$$\sigma_z = \frac{F}{\pi D t} = \frac{p D}{4 t}$$

环向: $F = p \cdot D b$

$$\sigma_\theta = \frac{F}{2 b t} = \frac{p D}{2 t}$$

环向 > 轴向

结论: $\sigma_\theta = 2 \sigma_z$

★ 截面核心

力矩作用点和中性轴在形心两侧

弯曲中心(剪切中心)

非对称弯曲产生

组合梁问题

中性轴在左较大一侧

弯曲切应力

扇形形心坐标

压杆稳定.

$$\lambda_{p^2} \pi \sqrt{\frac{E}{\sigma_p}}$$